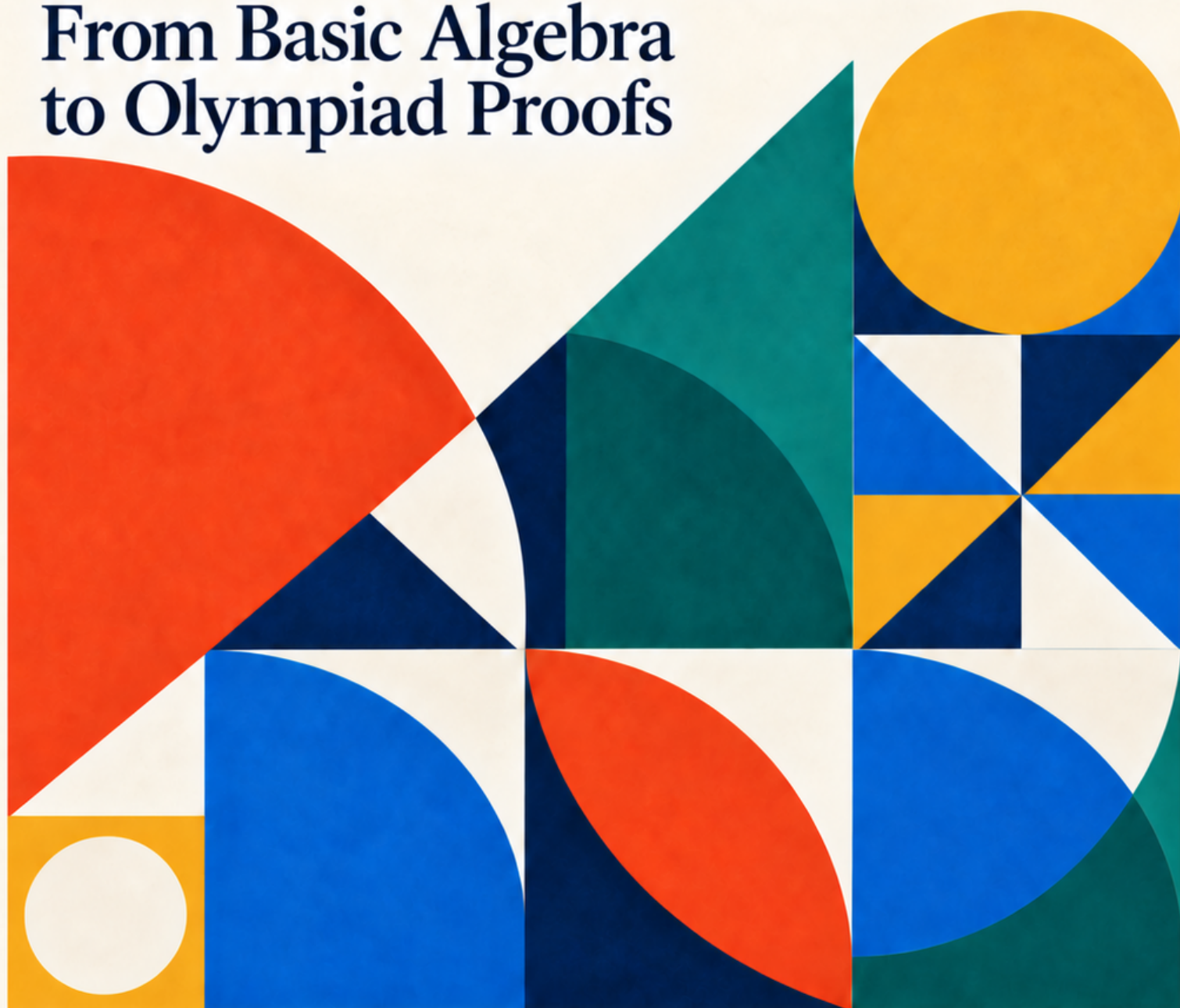


OLYMPIAD INEQUALITY

From Basic Algebra
to Olympiad Proofs



MATHIIT · STUDENT EDITION

Olympiad Inequality

From Basic Algebra to Olympiad Proofs

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12 main chapters · 4 advanced chapters

Worked examples · Graded exercises · Complete solutions

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CHAPTER 03

AM-GM and Classical Means

$$\frac{a+b}{2} \geq \sqrt{ab}$$

A short inequality can solve a long problem. The work is in choosing the right quantities and checking when they can be equal.

Before you begin

You need fractions, simple equations and the identity $(p - q)^2 = p^2 - 2pq + q^2$. Cube roots and the notation for a long sum are explained when they first appear.

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17 worked examples • **20** short checks • **37** exercises

01 · START WITH NUMBERS

What stays fixed?

The same sum can give very different products.

Two nonnegative numbers add to 10. Look at their products.

First number	Second number	Product
1	9	9
2	8	16
3	7	21
4	6	24
5	5	25

The product grows as the numbers get closer. The table suggests a rule. A proof must also cover pairs such as 4.7 and 5.3.

Why the equal pair wins

A square is never negative. For any nonnegative u, v ,

$$\begin{aligned}(u - v)^2 &\geq 0, \\ (u + v)^2 - 4uv &\geq 0, \\ uv &\leq \left(\frac{u + v}{2}\right)^2.\end{aligned}$$

If $u + v = 10$, this gives $uv \leq 25$. Equality means $(u - v)^2 = 0$, so $u = v = 5$.

THE TWO-TERM AM-GM INEQUALITY

The **arithmetic mean** is the usual average, $(u + v)/2$. The **geometric mean** is $\sqrt{uv}$. For $u, v \geq 0$,

$$\frac{u + v}{2} \geq \sqrt{uv}, \quad \text{or} \quad u + v \geq 2\sqrt{uv}.$$

Equality holds exactly when $u = v$. The initials give the name AM-GM.

YOUR TURN

T1. Find the arithmetic mean and the geometric mean of 4 and 9. Which is larger?

T2. Two nonnegative numbers have sum 18. What is their largest possible product? Give the numbers.

Write your steps first.

02 · CHOOSE THE INPUTS

A product that cancels

When one term contains x and another contains $1/x$, multiply them first.

WORKED EXAMPLE 1 A first minimum

For $x > 0$, find the least value of $x + 9/x$.

Solution. The terms x and $9/x$ are positive. Their product is always 9, so

$$x + \frac{9}{x} \geq 2\sqrt{x \cdot \frac{9}{x}} = 6.$$

To find out whether 6 is a minimum, check equality:

$$x = \frac{9}{x} \implies x^2 = 9 \implies x = 3.$$

At $x = 3$, the expression is $3 + 3 = 6$. **The minimum is 6, attained at $x = 3$.**

WORKED EXAMPLE 2 Keep the coefficients

For $x > 0$, find the least value of $2x + 18/x$.

Solution. This time the inputs are $2x$ and $18/x$, not x and $18/x$.

$$2x + \frac{18}{x} \geq 2\sqrt{2x \cdot \frac{18}{x}} = 2\sqrt{36} = 12.$$

Equality requires

$$2x = \frac{18}{x} \implies x^2 = 9 \implies x = 3.$$

Both inputs then equal 6. **The minimum is 12.**

What to notice. We did not guess a value of x first. We found a constant product, obtained a bound, and then solved the equality equation.

YOUR TURN

T3. For $x > 0$, find the least value of $x + 16/x$.

T4. For $x > 0$, find the least value of $3x + 12/x$.

Write your steps first.

03 · WORK IN THE OTHER DIRECTION

A sum that stays fixed

A fixed sum puts an upper bound on a product.

WORKED EXAMPLE 3 The largest rectangle

A rectangle has perimeter 20 cm. What is its greatest possible area?

Solution. Let its side lengths be $x, y > 0$. The perimeter gives $2(x + y) = 20$, so $x + y = 10$.

$$\text{Area} = xy \leq \left(\frac{x + y}{2}\right)^2 = 25.$$

Equality requires $x = y = 5$. Hence the greatest area is 25 cm^2 , attained by the square with side 5 cm.

WORKED EXAMPLE 4 Unequal variables, equal inputs

If $x, y > 0$ and $x + 3y = 12$, find the greatest value of xy .

Solution. The known sum is $x + 3y$. Use x and $3y$ as the two inputs:

$$x(3y) \leq \left(\frac{x + 3y}{2}\right)^2 = 36, \quad xy \leq 12.$$

Equality requires $x = 3y$. Together with $x + 3y = 12$, this gives

$$x = 6, \quad y = 2.$$

These values satisfy the condition and give $xy = 12$. **The maximum is 12.**

Equal inputs does not always mean equal variables. In Example 4, setting $x = y$ would give $x = y = 3$ and product 9, below the maximum.

YOUR TURN

T5. If $x, y > 0$ and $x + y = 14$, find the greatest value of xy .

T6. If $x, y > 0$ and $x + 2y = 16$, find the greatest value of xy .

Write your steps first.

04 · CHECK THE CLAIM

A bound is not always a minimum

The equality value must belong to the allowed domain.

WORKED EXAMPLE 5 An equality value that is forbidden

Find the least value of $x + 1/x$ when $x \geq 2$.

Solution. AM-GM gives $x + 1/x \geq 2$, but equality would require $x = 1$, which is forbidden. At the smallest allowed value, $x = 2$, the expression is $5/2$. Test this candidate by subtraction:

$$\begin{aligned} x + \frac{1}{x} - \frac{5}{2} &= \frac{2x^2 - 5x + 2}{2x} \\ &= \frac{(x-2)(2x-1)}{2x} \geq 0. \end{aligned}$$

The last step uses $x - 2 \geq 0$, $2x - 1 > 0$ and $2x > 0$. Equality holds at $x = 2$. **The minimum is $5/2$.**

WORKED EXAMPLE 6 A bound valid for negative numbers too

For every real t , prove

$$\frac{t}{1+t^2} \leq \frac{1}{2}.$$

Solution. Here t may be negative, so we should not use t as a positive AM-GM input. Instead, use a square:

$$\frac{1}{2} - \frac{t}{1+t^2} = \frac{1+t^2-2t}{2(1+t^2)} = \frac{(t-1)^2}{2(1+t^2)} \geq 0.$$

The denominator is positive for every real t . **Equality holds only at $t = 1$.**

YOUR TURN

T7. Find the least value of $x + 1/x$ when $x \geq 3$. Explain why the answer is not 2.

T8. A student claims that $x + 1/x \geq 2$ for every nonzero real x . Give a counterexample and correct the domain.

Write your steps first.

05 · ADD MORE INPUTS

Three terms, then n terms

The same principle survives: equal inputs give equality.

The cube root $\sqrt[3]{q}$ is the number whose cube is q . For instance, $\sqrt[3]{8} = 2$ because $2^3 = 8$.

AM-GM FOR THREE NONNEGATIVE NUMBERS

$$a + b + c \geq 3\sqrt[3]{abc}, \quad abc \leq \left(\frac{a + b + c}{3}\right)^3.$$

Equality holds exactly when $a = b = c$.

WORKED EXAMPLE 7 Three numbers with a fixed sum

If $a, b, c > 0$ and $a + b + c = 12$, find the greatest value of abc .

Solution. Use the product form of three-term AM-GM:

$$abc \leq \left(\frac{a + b + c}{3}\right)^3 = \left(\frac{12}{3}\right)^3 = 64.$$

Equality requires $a = b = c$. Their sum is 12, so each is 4. **The maximum is 64, attained at (4, 4, 4).**

The general form

For nonnegative numbers $a_1, a_2, \dots, a_n$,

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}.$$

The symbol $\sqrt[n]{P}$ means the nonnegative number whose n -th power is P . Equality holds exactly when all the inputs are equal.

We will use this with repeated inputs next. A proof of the general statement appears at the end of the chapter.

YOUR TURN

T9. If $a, b, c > 0$ and $abc = 8$, find the least value of $a + b + c$.

T10. If $a, b, c > 0$ and $a + b + c = 9$, find the greatest value of abc .

Write your steps first.

06 · MAKE THE POWERS MATCH

Split a term for a reason

Count how many copies of each variable the product needs.

WORKED EXAMPLE 8 A square in the product

If $x, y > 0$ and $x + y = 6$, find the greatest value of x^2y .

Solution. The product contains two copies of x and one of y . Split x into two equal parts so that the sum stays 6:

$$\underbrace{\frac{x}{2} + \frac{x}{2}}_x + y = 6.$$

Apply AM-GM to these three inputs:

$$\frac{x}{2} \cdot \frac{x}{2} \cdot y \leq \left(\frac{6}{3}\right)^3 = 8, \quad x^2y \leq 32.$$

Equality requires $x/2 = y = 2$, hence $(x, y) = (4, 2)$. **The maximum is 32.** Taking $x = y = 3$ would give only 27.

WORKED EXAMPLE 9 A square in the denominator

For $x > 0$, find the least value of $x + 4/x^2$.

Solution. Two copies of x will cancel x^2 in the denominator. Split the first term into halves:

$$\begin{aligned} x + \frac{4}{x^2} &= \frac{x}{2} + \frac{x}{2} + \frac{4}{x^2} \\ &\geq 3\sqrt[3]{\frac{x}{2} \cdot \frac{x}{2} \cdot \frac{4}{x^2}} = 3. \end{aligned}$$

Equality requires $x/2 = 4/x^2$, so $x^3 = 8$ and $x = 2$. All three inputs are then 1. **The minimum is 3.**

YOUR TURN

T11. If $x, y > 0$ and $x + y = 9$, find the greatest value of xy^2 .

T12. For $x > 0$, find the least value of $x + 32/x^2$.

Write your steps first.

07 · COMBINE SMALL BOUNDS

One estimate for each piece

Several simple inequalities can prove a much larger one.

WORKED EXAMPLE 10 Multiply three estimates

For positive x, y, z , prove

$$(x + y)(y + z)(z + x) \geq 8xyz.$$

Solution. Each factor is a sum of two positive numbers:

$$x + y \geq 2\sqrt{xy}, \quad y + z \geq 2\sqrt{yz}, \quad z + x \geq 2\sqrt{zx}.$$

Because both sides of every bound are positive, we may multiply:

$$(x + y)(y + z)(z + x) \geq 8\sqrt{x^2y^2z^2} = 8xyz.$$

All three equalities hold together exactly when $x = y = z$.

WORKED EXAMPLE 11 Expand, then pair

For positive x, y, z , prove

$$(x + y + z) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \geq 9.$$

Solution. Expand the product. Put reciprocal pairs next to each other:

$$\begin{aligned} (x + y + z) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) &= 3 + \left(\frac{x}{y} + \frac{y}{x} \right) \\ &\quad + \left(\frac{y}{z} + \frac{z}{y} \right) + \left(\frac{z}{x} + \frac{x}{z} \right). \end{aligned}$$

Each bracket has product 1, so each is at least 2. The expression is therefore at least $3 + 2 + 2 + 2 = 9$. Equality requires $x = y = z$.

YOUR TURN

T13. For positive a, b, c , prove $a/b + b/c + c/a \geq 3$. When is equality possible?

T14. If $a, b, c > 0$ and $abc = 1$, prove $(1 + a)(1 + b)(1 + c) \geq 8$.

Write your steps first.

08 · USE EQUALITY TO SOLVE AN EQUATION

When every term reaches its limit

A sum can reach the sum of its upper bounds only if every bound is an equality.

WORKED EXAMPLE 12 Two terms at their upper bounds

Find all real u, v satisfying $\frac{u}{1+u^2} + \frac{v}{1+v^2} = 1$.

Solution. Example 6 gives an upper bound of $1/2$ for each term. If either were smaller, the total would be less than 1. Both terms must equal $1/2$, so

$$u = 1, \quad v = 1.$$

Substitution gives $1/2 + 1/2 = 1$, so this is the only solution.

WORKED EXAMPLE 13 Different powers, the same bound

Find all real x satisfying

$$\frac{x}{1+x^2} + \frac{2x^2}{1+x^4} + \frac{3x^3}{1+x^6} = 3.$$

Solution. Apply Example 6 with $t = x$, then $t = x^2$, then $t = x^3$:

$$\frac{x}{1+x^2} \leq \frac{1}{2}, \quad \frac{2x^2}{1+x^4} \leq 1, \quad \frac{3x^3}{1+x^6} \leq \frac{3}{2}.$$

The upper bounds add to 3, exactly the required sum. Thus all must be equalities. The first already forces $x = 1$; this value also makes the other two equalities true. **The only solution is $x = 1$.**

Reading a longer sum

$$\sum_{k=1}^n \frac{kx^k}{1+x^{2k}} = \frac{x}{1+x^2} + \frac{2x^2}{1+x^4} + \cdots + \frac{nx^n}{1+x^{2n}}.$$

The symbol $\sum$ means “add the terms”; k runs from 1 to n . Adding $1 + 2 + \cdots + n$ forwards and backwards gives n pairs of $n + 1$. Hence $1 + 2 + \cdots + n = n(n + 1)/2$.

YOUR TURN

T15. Find all real u, v such that $\frac{2u}{1+u^2} + \frac{3v}{1+v^2} = \frac{5}{2}$.

T16. For real x , find the greatest value of $3x^3/(1+x^6)$.

Write your steps first.

09 · USE THE CONDITION FIRST

A fraction can hide a simpler variable

Before applying an inequality, make the given relation useful.

WORKED EXAMPLE 14 Remove two denominators

Find ab when positive numbers a, b satisfy

$$\frac{a}{1+a} + \frac{b}{1+b} = 1.$$

Solution. The denominators are positive, so multiply by $(1+a)(1+b)$:

$$\begin{aligned} a(1+b) + b(1+a) &= (1+a)(1+b), \\ a + b + 2ab &= 1 + a + b + ab, \\ ab &= 1. \end{aligned}$$

WORKED EXAMPLE 15 Use the fixed product

Under the condition in Example 14, find the least value of $a + b$.

Solution. We now know $ab = 1$, so

$$a + b \geq 2\sqrt{ab} = 2.$$

Equality requires $a = b = 1$. These values satisfy the original condition because $1/2 + 1/2 = 1$.

The minimum is 2.

For three fractions, try a substitution

If $a/(1+a) + b/(1+b) + c/(1+c) = 1$, name the fractions themselves:

$$u = \frac{a}{1+a}, \quad v = \frac{b}{1+b}, \quad w = \frac{c}{1+c}.$$

Now $u, v, w > 0$ and $u + v + w = 1$. The next two questions supply the algebra needed to turn this into a product bound. You will put the steps together in 2.7.

YOUR TURN

T17. If $a > 0$ and $u = a/(1+a)$, show that $0 < u < 1$ and express a in terms of u .

T18. For positive u, v, w with $u + v + w = 1$, prove

$$(1-u)(1-v)(1-w) \geq 8uvw.$$

Write your steps first.

10 · LESSON

The classical means and weighted AM-GM

For positive $x_1, \dots, x_n$, the harmonic, geometric, arithmetic and quadratic means are

$$H = \frac{n}{\sum \frac{1}{x_i}}, \quad G = \left(\prod x_i\right)^{\frac{1}{n}}, \quad A = \frac{\sum x_i}{n}, \quad Q = \sqrt{\frac{\sum x_i^2}{n}}.$$

They satisfy $H \leq G \leq A \leq Q$. AM-GM gives $G \leq A$. Applied to the reciprocals it gives $\frac{1}{H} \geq \frac{1}{G}$, hence $H \leq G$. The identity $Q^2 - A^2 = \frac{1}{n} \sum_i (x_i - A)^2 \geq 0$ gives $A \leq Q$, since both means are positive. Equality throughout holds when all inputs agree.

Weights allow different inputs to contribute different amounts. If $\lambda_i \geq 0$, $\sum \lambda_i = 1$, and the inputs are positive, weighted AM-GM states

$$\sum_i \lambda_i x_i \geq \prod_i x_i^{\lambda_i}.$$

For rational weights, repeat inputs. Weights $\frac{1}{3}, \frac{2}{3}$, for instance, mean using x, y, y in the ordinary three-term inequality. Real weights follow by approximating them with rational weights of sum 1 and using continuity. Zero-weight inputs may be omitted. Equality holds precisely when all positive-weight inputs agree.

To prove necessity, suppose two such inputs u, v differ. Choose $\delta > 0$ smaller than both their weights. Remove weight δ from each input and give weight 2δ to $\sqrt{uv}$. The weights still sum to 1. The weighted product is unchanged, because $u^\delta v^\delta = (\sqrt{uv})^{2\delta}$. The weighted sum falls strictly by

$$\delta(u + v - 2\sqrt{uv}) > 0.$$

The weighted AM-GM inequality, already proved for real weights, applies to the new list. Its weighted sum is still at least the unchanged weighted product. Hence the original weighted sum was strictly greater than that product. This proves the equality condition for real weights as well.

The classical means and weighted AM-GM

CONTINUED

The power means are

$$M_t = \left(\frac{1}{n} \sum x_i^t \right)^{\frac{1}{t}}$$

for $t \neq 0$, with $M_0 = G$. The general power-mean theorem says $M_r \leq M_s$ when $r < s$. It contains the classical chain at parameters $-1, 0, 1, 2$. Chapter 13 derives the positive-parameter comparison from convexity; the full range is derived below.

WORKED EXAMPLE 16 Match the weights to the powers

Positive x, y satisfy $x + y = 10$. Find the maximum of $x^2 y^3$.

Solution.

The powers 2 and 3 suggest five AM-GM inputs: two equal pieces from x and three equal pieces from y .

Use

$$\frac{x}{2}, \frac{x}{2}, \frac{y}{3}, \frac{y}{3}, \frac{y}{3}.$$

Their sum is $x + y = 10$.

Their product is $\frac{x^2 y^3}{(2^2 3^3)}$, and it is at most $(\frac{10}{5})^5 = 32$. Thus $x^2 y^3 \leq 108 \cdot 32 = 3456$.

Equality requires $\frac{x}{2} = \frac{y}{3} = 2$, so $x = 4, y = 6$. These satisfy the condition and attain the value.

Answer. Maximum 3456 at $x = 4, y = 6$.

YOUR TURN

T19.

Calculate the harmonic, geometric, arithmetic and quadratic means of 1, 4.

11 · LESSON

Bernoulli and Young

For $t \geq -1$ and a positive integer m , Bernoulli's inequality is

$$(1+t)^m \geq 1+mt.$$

The case $m = 1$ is an identity. If it holds for m , multiply by $1+t \geq 0$:

$$(1+t)^{m+1} \geq (1+mt)(1+t) = 1+(m+1)t+mt^2 \geq 1+(m+1)t.$$

Induction proves the statement. For m at least 2, equality holds only at $t = 0$. The case $m = 2$ has squared difference t^2 , and the induction step is strict for nonzero t .

For real exponents $\alpha \geq 1$, the same lower bound holds when $t > -1$; for $0 \leq \alpha \leq 1$ it reverses. These versions follow from convexity or concavity of powers, rather than from integer induction.

Young's inequality. If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for nonnegative u, v ,

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

For positive inputs this is weighted AM-GM on u^p, v^q , with weights $\frac{1}{p}, \frac{1}{q}$. Zero inputs are checked directly. Equality holds when $u^p = v^q$. The case $p = q = 2$ is the familiar bound $2uv \leq u^2 + v^2$.

Bernoulli and Young

CONTINUED

WORKED EXAMPLE 17 Young as repeated AM-GM

For u, v nonnegative, prove

$$uv \leq \frac{u^3}{3} + \frac{2v^{\frac{3}{2}}}{3}.$$

Solution.

The coefficients add to 1 and suggest weights $\frac{1}{3}$ and $\frac{2}{3}$.

Choose $p = 3$ and $q = \frac{3}{2}$. Then

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{3} + \frac{2}{3} = 1.$$

Young gives exactly

$$uv \leq \frac{u^3}{3} + \frac{v^{\frac{3}{2}}}{\frac{3}{2}} = \frac{u^3}{3} + \frac{2v^{\frac{3}{2}}}{3}.$$

Equivalently use AM-GM on $u^3, v^{\frac{3}{2}}, v^{\frac{3}{2}}$.

Equality means $u^3 = v^{\frac{3}{2}}$, which for nonnegative inputs is equivalent to $v = u^2$. The zero pair also fits.

Answer. Equality precisely when $v = u^2$.

YOUR TURN

T20.

Prove $(1+t)^3 \geq 1+3t$ for $t \geq -1$, and find equality.

EXERCISE · LEVEL-1

Building Fundamental

Use the appropriate AM-GM inputs and check equality.

- 1.1.** Find the arithmetic mean and geometric mean of 2 and 8. Verify the AM-GM inequality numerically.
- 1.2.** If $p, q > 0$ and $pq = 49$, find the least value of $p + q$.
- 1.3.** For $t > 0$, find the least value of $t + 25/t$.
- 1.4.** For $t > 0$, find the least value of $4t + 9/t$.
- 1.5.** If $p, q \geq 0$ and $p + q = 16$, find the greatest value of pq .
- 1.6.** If $x, y > 0$ and $2x + y = 18$, find the greatest value of xy .
- 1.7.** If $a, b > 0$ and $3a + 2b = 12$, find the greatest value of ab .
- 1.8.** A rectangle has perimeter 28 cm. Find its greatest possible area and the side lengths that give it.

- 1.9.** Find the least value of $x + 4/x$ for $x \geq 3$.
- 1.10.** A student applies two-term AM-GM directly to the real numbers t and $1/t$, without checking their signs. What is missing? Show that $t = -2$ makes the claimed bound $t + 1/t \geq 2$ false.
- 1.11.** For real t , prove $2t/(1 + t^2) \leq 1$ and determine equality.
- 1.12.** For $x > 0$, prove $1/(x + 1/x) \leq 1/2$ and determine equality.
- 1.13.** If $a, b, c > 0$ and $abc = 27$, find the least value of $a + b + c$.
- 1.14.** If $a, b, c > 0$ and $a + b + c = 15$, find the greatest value of abc .
- 1.15.** If $x, y > 0$ and $x + y = 12$, find the greatest value of x^2y .
- 1.16.** For $x > 0$, find the least value of $x + 108/x^2$.

Building Fundamental

CONTINUED

1.17. For positive a, b, c , prove $a/c + c/b + b/a \geq 3$ and determine equality.

1.18. For positive x, y, z , prove

$$(x + 2y)(y + 2z)(z + 2x) \geq 16\sqrt{2}xyz.$$

Can equality occur?

1.19. If $a, b, c > 0$ and $a + b + c = 6$, prove $1/a + 1/b + 1/c \geq 3/2$ and determine equality.

1.20. If $a, b, c > 0$ and $abc = 8$, prove $(2 + a)(2 + b)(2 + c) \geq 64$ and determine equality.

1.21. Write out $\sum_{k=1}^4 k/(1 + k)$ and evaluate it. The letter k takes the values 1, 2, 3, 4.

1.22. Find all real u, v such that

$$\frac{u}{1 + u^2} + \frac{4v}{1 + v^2} = \frac{5}{2}.$$

1.23. If $a, b > 0$ and $a/(1 + a) + b/(1 + b) = 1$, find the least value of $a^2 + b^2$.

1.24. Positive a, b, c satisfy

$$\frac{a}{1 + a} + \frac{b}{1 + b} + \frac{c}{1 + c} = 1.$$

Set $u = a/(1 + a)$, $v = b/(1 + b)$, $w = c/(1 + c)$. Express abc in terms of u, v, w , using only sums in the denominator.

1.25.

For positive a, b , prove

$$\frac{2ab}{a + b} \leq \sqrt{ab} \leq \frac{a + b}{2} \leq \sqrt{\frac{a^2 + b^2}{2}}.$$

1.26.

For t nonnegative and integer m at least 1, prove

$$(1 + t)^m \geq 1 + mt + \binom{m}{2}t^2.$$

1.27.

Positive x, y satisfy $x + 2y = 12$. Find the maximum of xy^2 .

1.28.

For nonnegative u, v prove

$$u^4 + 3v^{\frac{4}{3}} \geq 4uv.$$

EXERCISE · LEVEL-2

Olympiad

Choose the method and write a complete proof.

2.1. Positive numbers x, y, z satisfy $x + y + z = 6$. Find the greatest value of xy^2z^3 .

2.2. For $x > 0$, find the least value of

$$x + \frac{2}{x} + \frac{2}{x^2}.$$

2.3. Nonnegative numbers a, b, c satisfy $a + b + c = 3$. Find the greatest value of $(a + b)(b + c)(c + a)$.

2.4. Positive numbers a, b, c satisfy $abc = 1$. Prove

$$a^2 + b^2 + c^2 \geq \sqrt{a} + \sqrt{b} + \sqrt{c},$$

and determine equality.

2.5. Positive numbers x, y satisfy $x + y + xy = 3$. Find the greatest value of

$$F = x^2y^2(x + y).$$

Does F have a least value? Justify both conclusions.

2.6. For positive x, y, z , find the least value of

$$x + y + z + \frac{12}{xy} + \frac{30}{yz} + \frac{60}{zx}.$$

2.7. Positive real numbers a, b, c satisfy

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

Prove that $abc \leq 1/8$. Also determine when equality holds.

2.8. Let n be a positive integer. Find all real x for which

$$\sum_{k=1}^n \frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}.$$

2.9.

Let $a_1, \dots, a_n > 0$, where $n \geq 2$, and set $m = \min a_i$, $M = \max a_i$. Prove the quantitative form of AM-GM

$$\sum_{i=1}^n a_i - n(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \geq (\sqrt{M} - \sqrt{m})^2.$$

Determine equality, and show that the coefficient 1 on the right cannot be increased.

SOLUTIONS · LEVEL-1

Building Fundamental

The solution numbers match the exercise numbers.

1.1

$$\frac{2+8}{2} = 5, \quad \sqrt{2 \cdot 8} = 4.$$

Thus $5 \geq 4$. Equality does not hold because $2 \neq 8$.

1.2

$$p + q \geq 2\sqrt{pq} = 14.$$

Equality holds at $p = q = 7$, which satisfies the product condition. The minimum is 14.

1.3

$$t + \frac{25}{t} \geq 2\sqrt{25} = 10.$$

Equality requires $t = 25/t$, hence $t = 5$. The minimum is 10.

1.4

$$4t + \frac{9}{t} \geq 2\sqrt{4t \cdot \frac{9}{t}} = 12.$$

Equality requires $4t = 9/t$, so $t^2 = 9/4$. Since $t > 0$, $t = 3/2$. The minimum is 12.

1.5

$$pq \leq \left(\frac{p+q}{2}\right)^2 = 64.$$

Equality holds at $p = q = 8$, so the maximum is 64. Allowing zero does not change it.

1.6

Apply AM-GM to $2x$ and y :

$$2xy \leq \left(\frac{2x+y}{2}\right)^2 = 81, \quad xy \leq \frac{81}{2}.$$

Equality requires $2x = y = 9$, so $(x, y) = (9/2, 9)$.

Building Fundamental

CONTINUED

1.7

The inputs $3a$ and $2b$ have sum 12:

$$6ab \leq \left(\frac{12}{2}\right)^2 = 36, \quad ab \leq 6.$$

Equality holds when $3a = 2b = 6$, giving $(a, b) = (2, 3)$.

1.8

If the sides are $u, v > 0$, then $u + v = 14$. Therefore

$$uv \leq \left(\frac{14}{2}\right)^2 = 49.$$

Equality holds at $u = v = 7$. The greatest area is 49 cm^2 .

1.9

The AM-GM bound 4 has equality at $x = 2$, outside the domain. At $x = 3$, the expression is $13/3$. Prove this is the minimum:

$$x + \frac{4}{x} - \frac{13}{3} = \frac{(x-3)(3x-4)}{3x} \geq 0.$$

All relevant factors have the stated signs for $x \geq 3$; equality occurs at $x = 3$.

1.10

AM-GM requires nonnegative inputs. Since $1/t$ must be defined, this application requires $t > 0$. At $t = -2$,

$$t + \frac{1}{t} = -2 - \frac{1}{2} = -\frac{5}{2} < 2.$$

A nonzero denominator alone does not make an AM-GM application valid.

1.11

$$1 - \frac{2t}{1+t^2} = \frac{(t-1)^2}{1+t^2} \geq 0.$$

The denominator is always positive. Equality holds exactly when $t = 1$.

Building Fundamental

CONTINUED

1.12

AM-GM gives $x + 1/x \geq 2 > 0$. For positive numbers, taking reciprocals reverses the inequality:

$$\frac{1}{x + 1/x} \leq \frac{1}{2}.$$

Equality requires $x = 1/x$, hence $x = 1$.

1.13

$$a + b + c \geq 3\sqrt[3]{27} = 9.$$

Equality holds at $a = b = c = 3$; the minimum is 9.

1.14

$$abc \leq \left(\frac{15}{3}\right)^3 = 125.$$

Equality holds at $a = b = c = 5$, giving the maximum 125.

1.15

Use the inputs $x/2, x/2, y$, whose sum is 12:

$$\frac{x^2 y}{4} \leq \left(\frac{12}{3}\right)^3 = 64, \quad x^2 y \leq 256.$$

Equality requires $x/2 = y = 4$, so $(x, y) = (8, 4)$.

1.16

$$\frac{x}{2} + \frac{x}{2} + \frac{108}{x^2} \geq 3\sqrt[3]{\frac{x^2}{4} \cdot \frac{108}{x^2}} = 9.$$

Equality requires $x/2 = 108/x^2$, so $x^3 = 216$ and $x = 6$. All three inputs are 3. The minimum is 9.

1.17

The three positive inputs have product 1. Thus

$$\frac{a}{c} + \frac{c}{b} + \frac{b}{a} \geq 3.$$

At equality the three ratios are equal, so their common value has cube 1. It is 1, and $a = b = c$.

Building Fundamental

CONTINUED

1.18

$$\begin{aligned}x + 2y &\geq 2\sqrt{2xy}, \\y + 2z &\geq 2\sqrt{2yz}, \\z + 2x &\geq 2\sqrt{2zx}.\end{aligned}$$

Multiplying gives $8\sqrt{8x^2y^2z^2} = 16\sqrt{2}xyz$. Equality would require $x = 2y$, $y = 2z$, $z = 2x$, hence $x = 8x$, impossible for $x > 0$. Thus the displayed inequality is valid, but equality never occurs. It is a bound, not a claim of the best constant.

1.19

As in Example 11, expanding into three reciprocal pairs gives

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9.$$

Since the first factor is 6, division gives the required bound $3/2$. Equality holds at $a = b = c = 2$.

1.20

For each factor, $2 + a \geq 2\sqrt{2a}$, and similarly for b, c . Hence

$$(2 + a)(2 + b)(2 + c) \geq 8\sqrt{8abc} = 8\sqrt{64} = 64.$$

Equality holds exactly when $a = b = c = 2$, which satisfies $abc = 8$.

1.21

$$\sum_{k=1}^4 \frac{k}{1+k} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} = \frac{30 + 40 + 45 + 48}{60} = \frac{163}{60}.$$

1.22

The first term is at most $1/2$, and the second at most 2, by Example 6. The required total is the sum of these upper bounds. Both must be equalities, so $u = v = 1$. Substitution verifies the result.

1.23

Multiplying the condition by $(1 + a)(1 + b)$ gives $ab = 1$, as in Example 14. Therefore

$$a^2 + b^2 \geq 2\sqrt{a^2b^2} = 2ab = 2.$$

Equality requires $a = b$. Together with $ab = 1$ and positivity, this gives $a = b = 1$.

Building Fundamental

CONTINUED

1.24

We have $u + v + w = 1$, and T17 gives $a = u/(1 - u)$. Since $1 - u = v + w$,

$$a = \frac{u}{v + w}, \quad b = \frac{v}{w + u}, \quad c = \frac{w}{u + v}.$$

Multiplying gives

$$abc = \frac{uvw}{(v + w)(w + u)(u + v)}.$$

All denominator factors are positive.

1.25

$a + b \geq 2\sqrt{ab}$ by two-term AM-GM.

Multiplying the first target by $a + b$ reduces it to $2ab \leq (a + b)\sqrt{ab}$, which is the same AM-GM bound. For the last, the squared gap is $\frac{(a-b)^2}{4} \geq 0$.

All means are positive, so these divisions and squarings preserve equivalence. Each equality holds exactly when $a = b$.

Answer. The four means are ordered, with full equality exactly at $a = b$.

Building Fundamental

CONTINUED

1.26

The binomial theorem gives

$$(1+t)^m = \sum_{j=0}^m \binom{m}{j} t^j.$$

The terms $j = 0, 1, 2$ are the displayed right side, with $\binom{1}{2} = 0$.

All remaining terms are nonnegative since t is nonnegative and the coefficients are positive.

For $m = 1$ or 2 there are no remaining terms, so equality always holds. For m at least 3 , a positive cubic term exists when $t > 0$, so equality occurs only at $t = 0$.

Answer. The stated bound holds with the specified equality cases.

1.27

The inputs x, y, y have sum 12 , so their product is at most $(\frac{12}{3})^3 = 64$.

Equality requires $x = y = 4$, compatible with $x + 2y = 12$.

The coefficient 2 is already represented by the repeated y . Splitting y into halves would use the wrong sum.

Answer. Maximum 64 at $x = y = 4$.

1.28

Take u^4 and three copies of $v^{\frac{4}{3}}$. Their product is $u^4 v^4$.

The sum is at least four times the fourth root of that product, namely $4uv$. Nonnegative inputs are permitted.

All inputs must agree: $u^4 = v^{\frac{4}{3}}$, equivalent to $v = u^3$, including $u = v = 0$.

Answer. Equality exactly when $v = u^3$.

SOLUTIONS · LEVEL-2

Olympiad

The solution numbers match the exercise numbers.

2.1

The exponents tell us how many copies to create. Use

$$x, \quad \frac{y}{2}, \quad \frac{y}{2}, \quad \frac{z}{3}, \quad \frac{z}{3}, \quad \frac{z}{3}.$$

Their sum is $x + y + z = 6$, so six-term AM-GM gives

$$x \left(\frac{y}{2}\right)^2 \left(\frac{z}{3}\right)^3 \leq \left(\frac{6}{6}\right)^6 = 1.$$

Therefore $xy^2z^3 \leq 4 \cdot 27 = 108$. Equality requires all six inputs to be 1:

$$x = 1, \quad y = 2, \quad z = 3.$$

These numbers satisfy the condition and attain 108. **The maximum is 108.**

2.2

Finding the grouping. At $x = 1, 2, 4$, the expression is respectively $5, 7/2, 37/8$. This suggests testing $7/2$ as the minimum. At $x = 2$, the reciprocal terms are 1 and $1/2$. We want AM-GM inputs that match these values.

Pair $2/x$ with $x/2$, and group $2/x^2$ with two copies of $x/4$:

$$\begin{aligned} x + \frac{2}{x} + \frac{2}{x^2} &= \left(\frac{x}{2} + \frac{2}{x}\right) + \left(\frac{x}{4} + \frac{x}{4} + \frac{2}{x^2}\right) \\ &\geq 2\sqrt{\frac{x}{2} \cdot \frac{2}{x}} + 3\sqrt[3]{\frac{x}{4} \cdot \frac{x}{4} \cdot \frac{2}{x^2}} \\ &= 2 + \frac{3}{2} = \frac{7}{2}. \end{aligned}$$

Both groups reach equality at $x = 2$. Thus **the minimum is $7/2$** . The trial values suggested a candidate; the inequality proves it for every $x > 0$.

Olympiad

CONTINUED

2.3

The three inputs are nonnegative, and

$$(a + b) + (b + c) + (c + a) = 2(a + b + c) = 6.$$

Three-term AM-GM gives

$$(a + b)(b + c)(c + a) \leq \left(\frac{6}{3}\right)^3 = 8.$$

Equality requires $a + b = b + c = c + a = 2$. Subtracting pairs gives $a = b = c$, and the original sum gives $a = b = c = 1$. These values attain 8, so **the maximum is 8**.

2.4

The condition turns a product of two variables into a reciprocal: $ab = 1/c$, and similarly for the other pairs. This suggests the middle expression $ab + bc + ca$.

First, a sum of squares gives

$$a^2 + b^2 + c^2 - (ab + bc + ca) = \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{2} \geq 0.$$

Next, use two-term AM-GM:

$$\frac{1}{b} + \frac{1}{c} \geq \frac{2}{\sqrt{bc}} = 2\sqrt{a}.$$

The corresponding bounds for the other two pairs are

$$\frac{1}{c} + \frac{1}{a} \geq 2\sqrt{b}, \quad \frac{1}{a} + \frac{1}{b} \geq 2\sqrt{c}.$$

Add and divide by 2:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

Combining the two parts proves

$$a^2 + b^2 + c^2 \geq ab + bc + ca = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

Equality in the square identity forces $a = b = c$; the product condition then gives $a = b = c = 1$. This triple makes every step an equality.

Olympiad

CONTINUED

2.5

Reduce to one quantity. Put $s = x + y$. The condition gives $xy = 3 - s > 0$, so $s < 3$. Also

$$s^2 = (x + y)^2 \geq 4xy = 4(3 - s),$$

$$(s - 2)(s + 6) \geq 0.$$

Since $s > 0$, we obtain $s \geq 2$. Thus $2 \leq s < 3$.

Prove the upper bound. Now $F = s(3 - s)^2$. Write $u = s - 2$, so $0 \leq u < 1$. Then

$$F = (2 + u)(1 - u)^2 = 2 - 3u + u^3,$$

$$2 - F = u(3 - u^2) \geq 0.$$

Equality requires $u = 0$, hence $s = 2$ and $xy = 1$. These give $x = y = 1$. **The maximum is 2.**

Investigate the lower end. Positivity gives $F > 0$, so 0 is never attained. But take any $0 < t < 1$ and choose

$$x = t, \quad y = \frac{3 - t}{1 + t}.$$

Both are positive and $x + y + xy = t + y(1 + t) = 3$, so the condition holds. Moreover, $y < 3$ and $x + y < 4$, giving

$$0 < F = t^2 y^2 (t + y) < 36t^2.$$

By choosing positive t as small as needed, we make F smaller than any proposed positive minimum. Therefore **there is no least value**. The sharp lower bound is 0, approached but never reached.

Olympiad

CONTINUED

2.6

Choose a common equality point. Try $(4, 3, 5)$, where the expression is $4 + 3 + 5 + 1 + 2 + 3 = 18$. A proof needs a lower bound of 18 whose equality conditions all hold there.

Each denominator contains two variables. Group its fraction with one portion of each corresponding variable:

Fraction	Value at $(4, 3, 5)$	Matching portions
$12/(xy)$	1	$x/4, y/3$
$30/(yz)$	2	$2y/3, 2z/5$
$60/(zx)$	3	$3z/5, 3x/4$

The portions use exactly x, y, z : for example, $x/4 + 3x/4 = x$. The products inside the three groups are respectively 1, 8, 27. Hence

$$\begin{aligned}\frac{x}{4} + \frac{y}{3} + \frac{12}{xy} &\geq 3, \\ \frac{2y}{3} + \frac{2z}{5} + \frac{30}{yz} &\geq 6, \\ \frac{3z}{5} + \frac{3x}{4} + \frac{60}{zx} &\geq 9.\end{aligned}$$

Add the three inequalities to obtain the lower bound 18. All groups are equalities at $(4, 3, 5)$. Conversely, equality in the first group forces $x/4 = y/3 = 12/(xy) = 1$, so $x = 4, y = 3$; the second then forces $z = 5$. **The minimum is 18, attained only at $(4, 3, 5)$.**

The difficult step was choosing portions that share one equality point. Splitting each variable into arbitrary equal pieces would not ensure this.

Olympiad

CONTINUED

2.7

Make the condition a simple sum. Put

$$u = \frac{a}{1+a}, \quad v = \frac{b}{1+b}, \quad w = \frac{c}{1+c}.$$

Then $u, v, w > 0$ and $u + v + w = 1$. Solving for the original variables gives

$$a = \frac{u}{1-u} = \frac{u}{v+w}, \quad b = \frac{v}{w+u}, \quad c = \frac{w}{u+v}.$$

Therefore

$$abc = \frac{uvw}{(v+w)(w+u)(u+v)}.$$

Bound the denominator from below. By two-term AM-GM,

$$(v+w)(w+u)(u+v) \geq (2\sqrt{vw})(2\sqrt{wu})(2\sqrt{uv}) = 8uvw.$$

All quantities are positive, so

$$abc \leq \frac{uvw}{8uvw} = \frac{1}{8}.$$

Equality requires $u = v = w$; their sum gives $u = v = w = 1/3$. Thus

$$a = b = c = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}.$$

Substitution confirms the original condition and product $1/8$.

The olympiad step. The condition did not supply a product directly. Naming its fractions turned it into a sum, after which a familiar product inequality applied.

Olympiad

CONTINUED

2.8

Recognise the total. The right side is $\frac{1}{2}(1 + 2 + \cdots + n)$.

This suggests bounding term k by $k/2$.

For every real t ,

$$\frac{1}{2} - \frac{t}{1+t^2} = \frac{(t-1)^2}{2(1+t^2)} \geq 0.$$

Set $t = x^k$ and multiply by $k > 0$:

$$\frac{kx^k}{1+x^{2k}} \leq \frac{k}{2}.$$

Adding from $k = 1$ to n gives

$$\sum_{k=1}^n \frac{kx^k}{1+x^{2k}} \leq \frac{1}{2} \sum_{k=1}^n k = \frac{n(n+1)}{4}.$$

Use equality. Each term has a nonnegative gap from its bound. For the sum of the gaps to be zero, every gap must be zero: $x^k = 1$ for each k . Since $n \geq 1$, the term $k = 1$ is present and forces $x = 1$.

Finally, at $x = 1$,

$$\sum_{k=1}^n \frac{k}{2} = \frac{n(n+1)}{4}.$$

Thus **the only real solution is** $x = 1$, including when $n = 1$. Negative values of x were covered because the square identity holds for every real input.

Olympiad

CONTINUED

2.9

Reorder the variables so that $a_1 = m$ and $a_2 = M$. For $k \geq 2$, write

$$G_k = (a_1 \cdots a_k)^{\frac{1}{k}}, \quad D_k = \sum_{i=1}^k a_i - kG_k.$$

For $k \geq 3$, apply AM-GM to $k - 1$ copies of G_{k-1} and one copy of a_k . Their product is $a_1 \cdots a_k$, so

$$kG_k \leq (k-1)G_{k-1} + a_k.$$

Subtracting from the corresponding sums gives $D_k \geq D_{k-1}$. Therefore

$$D_n \geq D_{n-1} \geq \cdots \geq D_2 = m + M - 2\sqrt{mM} = (\sqrt{M} - \sqrt{m})^2.$$

Equality in a step requires $a_k = G_{k-1}$. Since $G_2 = \sqrt{mM}$, equality in every step holds exactly when all the remaining $n - 2$ entries are $\sqrt{mM}$. This includes the case in which all entries are equal. For $n = 2$, the identity is automatic.

When $m < M$, the tuple consisting of m, M and $n - 2$ copies of $\sqrt{mM}$ attains the bound with a nonzero right side. Hence a larger coefficient would fail.

Answer. Equality holds precisely for permutations of $(m, M, \sqrt{mM}, \dots, \sqrt{mM})$. For $n = 2$, every positive pair is an equality case. The coefficient 1 is best possible.

ANSWERS · YOUR TURN

Your turn

Compare the method as well as the answer.

T1

$$\text{Arithmetic mean} = \frac{4+9}{2} = \frac{13}{2}, \quad \text{geometric mean} = \sqrt{4 \cdot 9} = 6.$$

Since $13/2 = 6.5 > 6$, the arithmetic mean is larger. The two numbers are unequal, so equality in AM-GM is not expected.

T2

Call the numbers u, v . Then

$$uv \leq \left(\frac{u+v}{2} \right)^2 = 9^2 = 81.$$

Equality requires $u = v$. The sum is 18, so $u = v = 9$, which gives product 81.

T3

The two terms have product 16, so

$$x + \frac{16}{x} \geq 2\sqrt{16} = 8.$$

Equality requires $x = 16/x$. Thus $x^2 = 16$, and positivity gives $x = 4$. The minimum is 8.

T4

Use the terms $3x$ and $12/x$:

$$3x + \frac{12}{x} \geq 2\sqrt{3x \cdot \frac{12}{x}} = 12.$$

Equality means $3x = 12/x$, so $x = 2$. The minimum is 12.

T5

$$xy \leq \left(\frac{x+y}{2} \right)^2 = 49.$$

Equality is possible at $x = y = 7$, so the maximum is 49.

Your turn

CONTINUED

T6

The sum that is fixed belongs to the terms x and $2y$. Thus

$$x(2y) \leq \left(\frac{x+2y}{2}\right)^2 = 64, \quad xy \leq 32.$$

Equality requires $x = 2y = 8$, so $(x, y) = (8, 4)$.

T7

AM-GM gives the bound 2, but its equality value $x = 1$ is outside the domain. For $x \geq 3$,

$$x + \frac{1}{x} - \frac{10}{3} = \frac{(x-3)(3x-1)}{3x} \geq 0.$$

Hence the minimum is $10/3$, attained at $x = 3$.

T8

At $x = -1$, the left side is -2 , so the claim is false. The AM-GM statement applies here when $x > 0$. Both inputs x and $1/x$ are then positive.

T9

$$a + b + c \geq 3\sqrt[3]{abc} = 3\sqrt[3]{8} = 6.$$

Equality holds at $a = b = c = 2$, so the minimum is 6.

T10

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 = 27.$$

Equality holds at $a = b = c = 3$, giving the maximum 27.

T11

Use the three inputs $x, y/2, y/2$. Their sum is 9, so

$$\frac{xy^2}{4} \leq \left(\frac{x+y/2+y/2}{3}\right)^3 = 27.$$

Therefore $xy^2 \leq 108$. Equality requires $x = y/2 = 3$, so $(x, y) = (3, 6)$.

Your turn

CONTINUED

T12

The power x^2 in the denominator suggests two copies of x . Split the first term:

$$\frac{x}{2} + \frac{x}{2} + \frac{32}{x^2} \geq 3\sqrt[3]{\frac{x}{2} \cdot \frac{x}{2} \cdot \frac{32}{x^2}} = 6.$$

At $x = 4$, all three inputs are 2. Thus the minimum is 6.

T13

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3\sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} = 3.$$

At equality, the three ratios are equal and their product is 1. Each ratio is therefore 1, so $a = b = c$.

T14

Apply the two-term inequality to each factor and multiply the positive bounds:

$$(1+a)(1+b)(1+c) \geq (2\sqrt{a})(2\sqrt{b})(2\sqrt{c}) = 8\sqrt{abc} = 8.$$

All three bounds are equalities exactly when $a = b = c = 1$.

T15

For every real t , $t/(1+t^2) \leq 1/2$, with equality only at $t = 1$. Hence the two terms are at most 1 and $3/2$. Their sum reaches $5/2$ only when both bounds are equalities. Thus $u = v = 1$; substitution verifies the equation.

T16

Set $t = x^3$. Then $1 + x^6 = 1 + t^2 > 0$, and

$$\frac{3x^3}{1+x^6} = \frac{3t}{1+t^2} \leq \frac{3}{2}.$$

Equality requires $t = 1$, hence $x = 1$. This proves that $3/2$ is the maximum on the whole real line.

Your turn

CONTINUED

T17

Both a and $1 + a$ are positive, and $a < 1 + a$, so $0 < u < 1$. Now

$$u(1 + a) = a \implies u = a(1 - u) \implies a = \frac{u}{1 - u}.$$

The division is valid because $1 - u > 0$.

T18

Use the sum to replace each subtraction by a sum of positive terms:

$$\begin{aligned} (1 - u)(1 - v)(1 - w) &= (v + w)(w + u)(u + v) \\ &\geq (2\sqrt{vw})(2\sqrt{wu})(2\sqrt{uv}) \\ &= 8uvw. \end{aligned}$$

Equality holds when $u = v = w = 1/3$.

T19

$$H = \frac{2}{1 + 1/4} = \frac{8}{5}, \quad G = 2, \quad A = \frac{5}{2}, \quad Q = \sqrt{\frac{17}{2}}.$$

They occur in this increasing order.

T20

$$(1 + t)^3 - 1 - 3t = t^2(t + 3) \geq 0.$$

Since $t + 3 \geq 2$, equality occurs only at $t = 0$.

A CLOSER LOOK

Why AM-GM works for every n

An optional proof after you are comfortable using the inequality.

1. Double a number of inputs

The two-term case was proved in Lesson 1. Suppose AM-GM is known for m inputs. Split $2m$ nonnegative inputs into two groups, each with m inputs, and call their arithmetic means A, B .

The product within each group is at most the corresponding mean raised to the power m . Thus the product P of all $2m$ inputs satisfies

$$P \leq A^m B^m = (AB)^m \leq \left(\frac{A+B}{2} \right)^{2m}.$$

The last inequality is the two-term case applied to A, B , and $(A+B)/2$ is the mean of all $2m$ original inputs. So the result holds for $2m$ inputs.

Starting with 2, this proves AM-GM for 4, 8, 16, $\dots$ inputs.

2. Step down by one input

Suppose AM-GM is known for m inputs. Take $m-1$ nonnegative numbers with arithmetic mean A and product P . Append A as one more input. The mean of the new m inputs is still A , so

$$PA \leq A^m.$$

If $A > 0$, divide by A to get $P \leq A^{m-1}$, the desired result. If $A = 0$, all the original nonnegative inputs are zero and the result is immediate.

For any chosen n , start with a power of 2 at least as large as n , then step down until you reach n .

3. Track equality

For positive inputs, equality in the doubling step requires equality inside each group and $A = B$, hence all inputs equal. Equality in the downward step requires all the appended inputs equal, so the original inputs are equal too.

If some input is zero and some other input is positive, the geometric mean is zero but the arithmetic mean is positive. Equality is impossible in that case. If all inputs are zero, equality holds.

Therefore, for nonnegative inputs, equality holds exactly when all inputs are equal.

KEEP BESIDE YOUR WORKING

Choosing AM-GM well

The formula is short. These decisions do the work.

What you see	What to try	Go back to
A fixed product	Bound a sum from below.	Examples 1-2
A fixed sum	Bound a product from above.	Examples 3-4
Different powers	Repeat or split inputs to match the powers.	Examples 8-9
Several factors or reciprocal pairs	Prove smaller bounds and combine them.	Examples 10-11
An equation reaching a total bound	Force equality in every term.	Examples 12-13
A fractional condition	Simplify it or name the fractions.	Examples 14-15

Before you call the proof complete

1. Are all AM-GM inputs nonnegative? Are all denominators nonzero?
2. Did the sum or product of the chosen inputs simplify in a useful way?
3. Does taking a reciprocal reverse the direction?
4. Can all equality conditions hold at the same allowed values?
5. For a minimum or maximum, did you substitute those values to show the bound is attained?

What comes next

Cauchy–Schwarz will use a fixed sum of squares to control a linear expression, and turn positive denominator sums into reciprocal bounds.



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Olympiad

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16 chapters

140 worked examples

282 exercises



Try the 128 short checks as you read. Complete solutions explain the reasoning, the choice of method and every equality case.

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