Note:

1. gcd(a, b) denotes the greatest common divisor of integers a and b.
2. $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.
3. For a positive real number m, $\sqrt{m}$ denotes the positive square root of m. For example, $\sqrt{4} = +2$.
4. Unless otherwise stated all numbers are written in decimal notation.

Questions

1. The smallest positive integer that does not divide $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$ is:
2. The number of four-digit odd numbers having digits $1, 2, 3, 4$, each occuring exactly once, is:
3. The number obtained by taking the last two digits of 5^{2024} in the same order is:
4. Let $ABCD$ be a quadrilateral with $\angle ADC = 70°, \angle ACD = 70°$, $\angle ACB = 10°$ and $\angle BAD = 110°$. The measure of $\angle CAB$ (in degrees) is:
5. Let $a = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, let $b = \frac{x}{z} + \frac{y}{x} + \frac{z}{y}$ and let $c = \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right)$. The value of $|ab - c|$ is:

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6. Find the number of triples of real numbers (a, b, c) such that $a^{20} + b^{20} + c^{20} = a^{24} + b^{24} + c^{24} = 1$.

7. Determine the sum of all possible surface areas of a cube two of whose vertices are $(1, 2, 0)$ and $(3, 3, 2)$.

8. Let n be the smallest integer such that the sum of digits of n is divisible by 5 as well as the sum of digits of $(n + 1)$ is divisible by 5. What are the first two digits of n in the same order?

9. Consider the grid of points $X = \{(m, n) \mid 0 \leq m, n \leq 4\}$. We say a pair of points $\{(a, b), (c, d)\}$ in X is a knight-move pair if ($c = a \pm 2$ and $d = b \pm 1$) or ($c = a \pm 1$ and $d = b \pm 2$). The number of knight-move pairs in X is:

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10. Determine the number of positive integral values of p for which there exists a triangle with sides a, b, and c which satisfy

$$a^2 + (p^2 + 9)b^2 + 9c^2 - 6ab - 6pbc = 0.$$

11. The positive real numbers a, b, c satisfy:

$$\frac{a}{2b+1} + \frac{2b}{3c+1} + \frac{3c}{a+1} = 1$$
$$\frac{1}{a+1} + \frac{1}{2b+1} + \frac{1}{3c+1} = 2$$

What is the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$?

12. Consider a square $ABCD$ of side length 16. Let E, F be points on CD such that $CE = EF = FD$. Let the line BF and AE meet in M. The area of $\triangle MAB$ is:

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13. Three positive integers a, b, c with $a > c$ satisfy the folowing equations:

$$ac + b + c = bc + a + 66, \quad a + b + c = 32.$$

Find the value of a.

14. Initially, there are 3^{80} particles at the origin $(0,0)$. At each step the particles are moved to points above the x-axis as follows: if there are n particles at any point (x,y), then $\left\lfloor \frac{n}{3} \right\rfloor$ of them are moved to $(x+1, y+1)$, $\left\lfloor \frac{n}{3} \right\rfloor$ are moved to $(x, y+1)$ and the remaining to $(x-1, y+1)$. For example, after the first step, there are 3^{79} particles each at $(1,1)$, $(0,1)$ and $(-1,1)$. After the second step, there are 3^{78} particles each at $(-2,2)$ and $(2,2)$, 2×3^{78} particles each at $(-1,2)$ and $(1,2)$, and 3^{79} particles at $(0,2)$. After 80 steps, the number of particles at $(79,80)$ is:

15. Let X be the set consisting of twenty positive integers $n, n+2, \ldots, n+38$. The smallest value of n for which any three numbers $a, b, c \in X$, not necessarily distinct, form the sides of an acute-angled triangle is:

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16. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the relation $4f(3-x) + 3f(x) = x^2$ for any real x. Find the value of $f(27) - f(25)$ to the nearest integer. (Here $\mathbb{R}$ denotes the set of real numbers.)

17. Consider an isosceles triangle ABC with sides $BC = 30$, $CA = AB = 20$. Let D be the foot of the perpendicular from A to BC, and let M be the midpoint of AD. Let PQ be a chord of the circumcircle of triangle ABC, such that M lies on PQ and PQ is parallel to BC. The length of PQ is:

18. Let p, q be two-digit numbers neither of which are divisible by 10. Let r be the four-digit number by putting the digits of p followed by the digits of q (in order). As p, q vary, a computer prints r on the screen if gcd$(p, q) = 1$ and $p + q$ divides r. Suppose that the largest number that is printed by the computer is N. Determine the number formed by the last two digits of N (in the same order).

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19. Consider five points in the plane, with no three of them collinear. Every pair of points among them is joined by a line. In how many ways can we color these lines by red or blue, so that no three of the points form a triangle with lines of the same color.

20. On a natural number n you are allowed two operations: (1) multiply n by 2 or (2) subtract 3 from n. For example starting with 8 you can reach 13 as follows: $8 \to 16 \to 13$. You need two steps and you cannot do in less than two steps. Starting from 11, what is the least number of steps required to reach 121?

21. An intenger n is such that $\left\lfloor \frac{n}{9} \right\rfloor$ is a three digit number with equal digits, and $\left\lfloor \frac{n-172}{4} \right\rfloor$ is a 4 digit number with the digits $2, 0, 2, 4$ in some order. What is the remainder when n is divided by 100?

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22. In a triangle ABC, $\angle BAC = 90°$. Let D be the point on BC such that $AB + BD = AC + CD$. Suppose $BD : DC = 2 : 1$. If $\frac{AC}{AB} = \frac{m + \sqrt{p}}{n}$, where m, n are relatively prime positive integers and p is a prime number, determine the value of $m + n + p$.

23. Consider the fourteen numbers, $1^4, 2^4, \ldots, 14^4$. The smallest natural number n such that they leave distinct remainders when divided by n is:

24. Consider the set F of all polynomials whose coefficients are in the set of $\{0, 1\}$. Let $q(x) = x^3 + x + 1$. The number of polynomials $p(x)$ in F of degree 14 such that the product $p(x)q(x)$ is also in F is:

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25. A finite set M of positive integers consists of distinct perfect squares and the number 92. The average of the numbers in M is 85. If we remove 92 from M, the average drops to 84. If N^2 is the largest possible square in M, what is the value of N?

26. The sum of $\lfloor x \rfloor$ for all real numbers x satisfying the equation $16 + 15x + 15x^2 = \lfloor x \rfloor^3$ is:

27. In a triangle ABC, a point P in the interior of ABC is such that

$$\angle BPC - \angle BAC = \angle CPA - \angle CBA = \angle APB - \angle ACB.$$

Suppose $\angle BAC = 30°$ and $AP = 12$. Let D, E, F be the feet of perpendiculars form P on to BC, CA, AB respectively. If $m\sqrt{n}$ is the area of the triangle DEF where m, n are integers with n prime, then what is the value of the product mn?

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28. Find the largest positive integer $n < 30$ such that $\frac{1}{2}(n^8 + 3n^4 - 4)$ is not divisible by the square of any prime number.

29. Let $n = 2^{19}3^{12}$. Let M denote the number of positive divisors of n^2 which are less than n but would not divide n. What is the number formed by taking the last two digits of M (in the same order)?

30. Let ABC be a right-angled triangle with $\angle B = 90°$. Let the length of the altitude BD be equal to 12. What is the minimum possible length of AC, given that AC and the perimeter of triangle ABC are integers?

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