RMO 2024

Official Solutions

Problem 1. Let $n > 1$ be a positive integer. Call a rearrangement $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ *nice* if for every $k = 2, 3, \ldots, n$, we have that $a_1 + a_2 + \cdots + a_k$ is **not** divisible by k.

(a) If $n > 1$ is odd, prove that there is no *nice* rearrangement of $1, 2, \ldots, n$.

(b) If n is even, find a *nice* rearrangement of $1, 2, \ldots, n$.

Solution. For the first part, note that the given condition for $k = n$ implies that the sum $a_1 + a_2 + \ldots + a_n$ is not divisible by n. However, $a_1, a_2, \ldots, a_n$ is a rearrangement of $1, 2, \ldots, n$ so their sum is equal to $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$ which is divisible by n for odd n. Thus, there cannot be any nice rearrangement of $1, 2, \ldots, n$ for odd n.

For the second part, let $n = 2m$. We show that the sequence

$$2, 1, 4, 3, 6, 5, 8, 7, \ldots\ldots, 2m, 2m-1$$

is a nice rearrangement of $1, 2, \ldots, 2m$. For k even, we have $a_1 + a_2 + \ldots + a_k = \dfrac{k(k+1)}{2}$ which is not divisible by k since $(k+1)/2$ is not an integer. For k odd, we have $a_1 + a_2 + \ldots + a_k = \dfrac{k(k+1)}{2} + 1$ which is 1 more than a multiple of k, so it is again not divisible by k for $k > 1$. □

Problem 2. For a positive integer n, let $R(n)$ be the sum of the remainders when n is divided by $1, 2, \ldots, n$. For example, $R(4) = 0 + 0 + 1 + 0 = 1$, $R(7) = 0 + 1 + 1 + 3 + 2 + 1 + 0 = 8$. Find all positive integers n such that $R(n) = n - 1$.

Solution. Let $n > 8$. The remainder when n is divided by some i satisfying $\frac{n}{2} < i \leq n$ is $(n - i)$. Adding, we get that

$$n - 1 = R(n) \geq \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} (n - i) = \sum_{k=1}^{\lceil \frac{n}{2} \rceil - 1} k = \frac{1}{2} \left\lceil \frac{n}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right) \geq \frac{1}{2} \left\lceil \frac{n}{2} \right\rceil \cdot 4 \geq n$$

This is a contradiction. So, we get that $n \leq 8$. Now we can compute that $R(1) = R(2) = 0, R(3) = R(4) = 1, R(5) = 4, R(6) = 3, R(7) = R(8) = 8$. Therefore, the only solutions are $n = 1$ and $n = 5$. □

Problem 3. Let ABC be an acute triangle with $AB = AC$. Let D be the point on BC such that AD is perpendicular to BC. Let O, H, G be the circumcentre, orthocentre and centroid of triangle ABC respectively. Suppose that $2 \cdot OD = 23 \cdot HD$. Prove that G lies on the incircle of triangle ABC.

Solution. Let I be the incenter of $\triangle ABC$. First note that O, G, H, I all lie on AD since it is simultaneously the perpendicular bisector of BC, the $A-$altitude, the $A-$ median and the angle bisector of $\angle BAC$.

Suppose the reflection of H across BC is M. Then M lies on the circumcircle of $\triangle ABC$ as well as lies on the angle bisector of $\angle BAC$, so it is the midpoint of arc BC not containing A. Then, we note that $\angle MBI = \angle MIB$, so $MB = MI$. Combining with $MB = MC$, we have that M is the circumcenter of $\triangle BIC$.

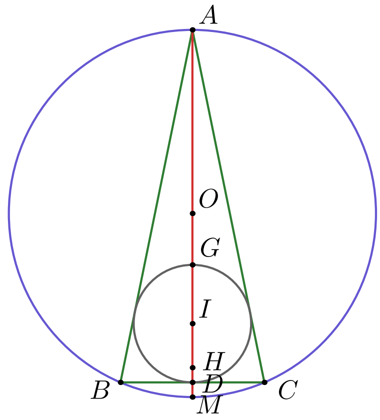
Now, let the circumradius of $\triangle ABC$ be R, let $OD = x$, $HD = y$. Then we have $x = \dfrac{23}{2}y$. Also, $R = OM = OD + DM = OD + HD = x + y$. Thus, $y = \dfrac{2}{25}R$. This implies that

$AD = 2R - y = \dfrac{48}{25}R$. Now, recall that G divides AD in the ratio $2:1$, so $GD = \dfrac{16}{25}R$.

Also, we have $\triangle MDB \sim \triangle MBA$ since the angle at M is common and $\angle MBD = \angle MAB$, both equalling $\angle BAC/2$. Therefore, $MB^2 = MD \cdot MA$, and hence

$$MI^2 = MD \cdot MA = y \cdot 2R = \frac{4}{25}R^2 \implies MI = \frac{2}{5}R.$$

Thus, $ID = \dfrac{8}{25}R$, which combined with $GD = \dfrac{16}{25}R$ implies that $GI = ID$ is equal to the inradius, proving that G lies on the incircle. □



Remark. A student well-versed in trigonometry may readily obtain $\cos A = 23/25$ by observing that $OD = R\cos A$ and $HD = 2R\cos B\cos C = 2R\cos^2(90^\circ - A/2) = R(1 - \cos A)$. Now $GD = AD/3 = (OA+OD)/3 = 16R/25$ and $r = AD/(1+\csc(A/2)) = 48R/(25\times 6) = 8R/25$ whence $GI = ID = r$ and the conclusion follows.

Problem 4. Let a_1, a_2, a_3, a_4 be real numbers such that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$. Show that there exist i, j with $1 \le i < j \le 4$, such that $(a_i - a_j)^2 \le \dfrac{1}{5}$.

Solution 1. Let m be the minimum of $|a_i - a_j|$ over all $1 \le i < j \le 4$. Without loss of generality, we may assume that $a_1 \le a_2 \le a_3 \le a_4$. Then $a_j - a_i \ge (j-i)m$ for all $1 \le i < j \le 4$. Thus,

$$\sum_{1\le i<j\le 4} (a_i - a_j)^2 \ge \sum_{1\le i<j\le 4} (j-i)^2 m^2 = 20m^2.$$

On the other hand,

$$\sum_{1\le i<j\le 4} (a_i - a_j)^2 = 4(a_1^2 + a_2^2 + a_3^2 + a_4^2) - (a_1 + a_2 + a_3 + a_4)^2 \le 4.$$

Thus, $20m^2 \le 4 \implies m^2 \le 1/5$. □

Solution 2. Suppose $|a_i - a_j| > \dfrac{1}{\sqrt{5}}$ for all $1 \le i < j \le 4$. Then if x, y are respectively the maximum and minimum among the a_i, then $x - y > \dfrac{3}{\sqrt{5}}$. Suppose u, v are the other two a_i apart from x, y. Then using $a^2 + b^2 \ge \dfrac{1}{2}(a-b)^2$, we have that

$$1 = x^2 + y^2 + u^2 + v^2 \ge \frac{1}{2}(x-y)^2 + \frac{1}{2}(u-v)^2 > \frac{1}{2}\left(\frac{9}{5} + \frac{1}{5}\right) = 1$$

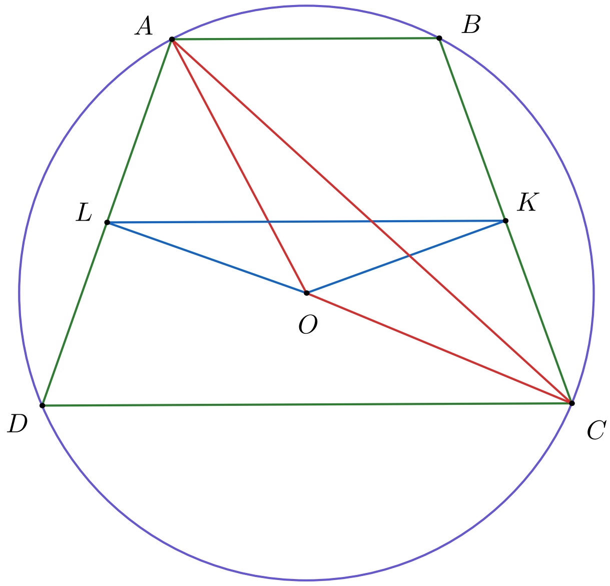
which is a contradiction. □

Remark. There is another solution involving casework where the cases involving the number of positive and negative a_i are distinguished. We exclude it for brevity.

Problem 5. Let $ABCD$ be a cyclic quadrilateral such that AB is parallel to CD. Let O be the circumcentre of $ABCD$, and L be the point on AD such that OL is perpendicular to AD. Prove that

$$OB \cdot (AB + CD) = OL \cdot (AC + BD).$$

Solution 1. Let K be the foot of perpendicular from O onto BC. Note that $ABCD$ is a isosceles trapezium, therefore $AC + BD = 2AC$. We have that L and K are the midpoints of AD and BC respectively, therefore $LK = (AB + CD)/2$. Also $OB = OA$. Thus it suffices to prove that $\dfrac{OA}{AC} = \dfrac{OL}{LK}$.



Now $\angle AOL = \angle ACD = \angle BDC = \angle COK$. Thus, $\angle AOC = \angle LOK$. Also note that $OL = OK$ since distance from center to two equal chords is the same. Thus, $\triangle AOC$ and $\triangle LOK$ are isosceles triangles with $\angle AOC = \angle LOK$, hence they are similar, which immediately implies the desired. $\square$

Solution 2. As before, note that $ABCD$ is an isosceles trapezium. Let the intersection of AC and BD be E, the foot of perpendicular from A onto CD be P, and let $AP = h$. Let $\angle BDC = \angle ACD = x$ Then $\angle BEC = 2x$. Let the radius $OB = R$. Thus, $[ABCD] = \frac{1}{2}AC^2 \sin 2x = \frac{1}{2}(AB + CD) \cdot h$. Now, note that $OL = R\cos x$ by considering $\triangle AOL$, and $h = AC \sin x$. Therefore, $(AB + CD) \cdot h = AC^2 \sin 2x = h \cdot 2 \cdot AC \cos x$. Hence

$$\frac{OL}{OB} = \cos x = \frac{AB + CD}{2AC}$$

which finishes the problem since $AC = BD$.

Problem 6. Let $n \geq 2$ be a positive integer. Call a sequence $a_1, a_2, \cdots, a_k$ of integers an *n-chain* if $1 = a_1 < a_2 < \cdots < a_k = n$, and a_i divides a_{i+1} for all $i, 1 \leq i \leq k-1$. Let $f(n)$ be the number of *n-chains* where $n \geq 2$. For example, $f(4) = 2$ corresponding to the 4-chains $\{1, 4\}$ and $\{1, 2, 4\}$.

Prove that $f(2^m \cdot 3) = 2^{m-1}(m+2)$ for every positive integer m.

Solution. We will prove that for any two distinct primes p, q, that $f(p^m \cdot q) = 2^{m-1}(m+2)$ for all integers $m \geq 1$. Suppose $n = p^m \cdot q$, and let $\{a_1, a_2, \cdots, a_k\}$ be a n-chain. Then a_i divides a_{i+1} implies that $a_{i+1}/a_i = p^{b_i} \cdot q^{c_i}$, where b_i, c_i are non-negative integers for $i = 1, \cdots, k-1$. Note that $a_{i+1} > a_i$ implies that b_i and c_i cannot be simultaneously 0.

Now, we have $b_1 + \ldots + b_{k-1} = m$ and $c_1 + \ldots + c_{k-1} = 1$. Thus, exactly one of the c_i will be equal to 1, and that implies that at most one of the b_i can be 0.

Recall that a *composition* of m is a sequence of positive integers adding to m. Corresponding to any l-length composition $x_1, \ldots, x_l$ of m, we will get exactly $2l + 1$ many n-chains. l of them are obtained by setting $b_i = x_i$ for all i and choosing one of $c_1, \ldots c_l$ to be 1, and rest to be 0. The other $l+1$ chains of length $l+1$ are obtained by choosing some $1 \leq j \leq l+1$, then setting $c_j = 1, b_j = 0$, $b_i = x_i$ for all $i < j$, and $b_i = x_{i-1}$ for all $i > j$.

This can be done in various ways as follows:

First way: it is well known that there are $\binom{m-1}{l-1}$ compositions of m with l parts. Therefore, we need

$$\begin{aligned}\sum_{l=1}^{m}\binom{m-1}{l-1}(2l+1) &= \sum_{l=0}^{m-1}\binom{m-1}{l}(2l+3)\\ &= 2\sum_{l=0}^{m-1} l\cdot\binom{m-1}{l} + 3\sum_{l=0}^{m-1}\binom{m-1}{l}\\ &= 2\left(\sum_{l=1}^{m-1}(m-1)\cdot\binom{m-2}{l-1}\right) + 3\cdot 2^{m-1}\\ &= 2(m-1)2^{m-2} + 3\cdot 2^{m-1} = 2^{m-1}(m+2).\end{aligned}$$

Second way: We will show that the total number of compositions of m is 2^{m-1} and the sum of the number of parts over all compositions of m is $2^{m-2}(m+1)$ via direct bijections. This finishes the problem, since we get the sum of $(2l+1)$ over all compositions to be $2\cdot 2^{m-2}(m+1) + 2^{m-1} = 2^{m-1}(m+2)$.

For the first one, consider sequences of 0's and 1's such that there are exactly m 1's, no two 0's are adjacent and the sequence begins and ends with a 1. Then we can choose whether or not to insert a 0 in the $m-1$ spaces between the 1's, hence there are 2^{m-1} possible ways to do it.

For the second one, we consider the above sequences but we put a single 0 at the end, and we also select a *special* 0. Then we can choose the special 0 first. If this is the last 0 then we get 2^{m-1} choices for the other zeroes, and if not then we have $m-1$ choices for the special 0, and then 2^{m-2} choices for the other spaces. This totals to $2^{m-2}(m+1)$ ways. □

Remark: There are other solutions involving induction using recursions of the form

$$f(2^m\cdot 3) = \sum_{0\le l<m} f(2^l\cdot 3) + \sum_{0\le l\le m} f(2^l)$$

or $f(2^m\cdot 3) = 2(f(2^{m-1}\cdot 3) + f(2^m) - f(2^{m-1}))$. Again, we omit them for the sake of brevity.